Efficient Card-based Protocols for Generating a Hidden Random Permutation without Fixed Points*

Rie Ishikawa[1], Eikoh Chida[1], and Takaaki Mizuki[2]

[1] Electrical and Computer Engineering,
National Institute of Technology, Ichinoseki College,
Takanashi, Hagisho, Ichinoseki 021–8511, Japan
g10205[atmark]g.ichinoseki.ac.jp
chida[atmark]g.ichinoseki.ac.jp
[2] Cyberscience Center, Tohoku University,
6–3 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980–8578, Japan
tm-paper+cardpermw[atmark]g-mail.tohoku-university.jp

Abstract. Consider the holiday season, where there are n players who would like to exchange gifts. That is, we would like to generate a random permutation having no fixed point. It is known that such a random permutation can be obtained in a hidden form by using a number of physical cards of four colors with identical backs, guaranteeing that it has no fixed point (without revealing the permutation itself). This paper deals with such a problem and improves the known result: whereas the known protocol needs $O(n^2)$ cards of four colors, our efficient protocol uses only $O(n \log n)$ cards of two colors.

1 Introduction

Consider the holiday season, where there are n players who would like to exchange gifts. We wish to avoid the undesirable situation in which a player must buy a present for himself/herself. That is, we need to produce a random permutation $\pi \in S_n$ that has no fixed point, where S_n denotes the symmetric group of degree n (throughout this paper). There is an unconventional solution to the "no fixed point" problem, i.e., it is known that such a random permutation can be obtained in a hidden form by using a number of physical cards of four colors, say $\boxed{\clubsuit}$, $\boxed{\heartsuit}$, $\boxed{\diamondsuit}$, and $\boxed{\spadesuit}$,[1] with identical backs $\boxed{?}$, guaranteeing that it has no fixed point (without revealing the permutation itself) [3]. This paper deals with such a problem and proposes an efficient approach that improves the known result.

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[1] Throughout this paper, we say that a card has the same "color" as another one if they have the same pattern on their face sides.

1.1 Known Method for Generating a Random Permutation

In 1993, Crépeau and Kilian gave a card-based protocol for generating a random permutation $\pi \in S_n$ without any fixed point [3]. Their protocol produces a pile of n cards that consists of $(n-1)$ $\boxed{\clubsuit}$s and one $\boxed{\heartsuit}$ with their faces down (on the table) for every player $p_i, 1 \leq i \leq n$:

$$p_i : \boxed{?}\boxed{?} \cdots \boxed{?} \cdots \boxed{?}.$$

The position of card $\boxed{\heartsuit}$ corresponds to the value of $\pi(i)$ when all the n cards are revealed:

$$p_i : \overset{1}{\boxed{\clubsuit}}\overset{2}{\boxed{\clubsuit}} \cdots \overset{\pi(i)}{\boxed{\heartsuit}} \cdots \overset{n}{\boxed{\clubsuit}}.$$

Thus, if player p_i looks at his/her pile privately, then the information about who p_i is going to buy a present for will be kept secret.

Because the protocol produces a pile of such cards for each of the n players, as seen above, it uses $n(n-1)$ $\boxed{\clubsuit}$s and n $\boxed{\heartsuit}$s. In addition, it requires a number of cards of different colors, namely $n^2/2$ $\boxed{\diamondsuit}$s and $n^2/2$ $\boxed{\spadesuit}$s. Thus, the known method needs $2n^2$ cards of four colors in total[2]. Further details are given in Section 2.

1.2 Our Results and Related Work

Table 1 summarizes both the known result and our results. As mentioned above, to generate a random permutation without fixed points, the known method [3] requires $2n^2$ cards of four colors. In this paper, we reduce the number of required colors and cards. First, we devise a new shuffling operation called a "pile-scramble shuffle" in Section 3. Using this new shuffle, we can enhance the efficiency of the known protocol, and consequently, we can show that n^2 cards of two colors are sufficient. We then show in Section 4 that $(2n\lceil \log n \rceil + 6)$ cards[3] of two colors are sufficient to solve the "no fixed point" problem by considering another expression of each player's index.

	No. of colors	No. of cards
Known protocol [3] (§2)	4	$2n^2$
Improvement with pile-scramble shuffle (§3)	2	n^2
Our main protocol (§4)	2	$2n\lceil \log n \rceil + 6$

Table 1. Performance of each protocol

[2] Note that we cannot use a standard deck of playing cards because each of them has a unique pattern on its face side.

[3] All logarithms are base 2 throughout this paper.

Before presenting our protocols, we present a complete description of the known protocol [3] in Section 2. Section 5 concludes this paper with some discussion.

Card-based cryptography allows us not only to generate a random permutation, but also to have various kinds of cryptographic protocols such as secure multiparty computations and zero-knowledge proof. For example, there are known protocols for securely computing AND [1, 3, 7, 8, 10, 13], XOR [3, 8, 9], adder [6], 3-variable symmetric functions [12], and so on. Furthermore, the relationship between playing cards and cryptography has been explored in the literature (e.g., [2, 4, 5, 14]).

2 Known Protocol

In this section, we present a complete description of the Crépeau-Kilian protocol [3] that generates a hidden random permutation having no fixed point.

Assume that n players $p_1, p_2, \ldots, p_n$ would like to produce a random permutation $\pi \in S_n$ without any fixed point. Their protocol consists of two phases, the Random-Permutation Generating phase and the Fixed-Point Checking phase, as follows.

[Random-Permutation Generating phase]

1-1. Using $n(n-1)$ [♣]s and n [♡]s, arrange the cards as below (putting each [♡] on the diagonal), and insert a "marker" after each row, where a marker consists of $n/2$ [♢]s and $n/2$ [♠]s (for simplicity, n is assumed to be an even number):

[♡][♣] ··· [♣] ··· [♣][♣][♢][♢][♢] ··· [♠][♠][♠]
[♣][♡] ··· [♣] ··· [♣][♣][♢][♢][♢] ··· [♠][♠][♠]
⋮
[♣][♣] ··· [♡] ··· [♣][♣][♢][♢][♢] ··· [♠][♠][♠]
⋮
[♣][♣] ··· [♣] ··· [♣][♡][♢][♢][♢] ··· [♠][♠][♠].

1-2. Turn over the cards so that they are all face down, and apply a random cut, i.e., a cyclic shuffle, to the sequence of $2n^2$ cards (obtained by row-wise concatenation).

1-3. Reveal the first card. If the face-up card is either [♣] or [♡], go back to step (1-2). If it is either [♢] or [♠], i.e., a marker, then proceed to the next step. Note that the probability of returning to step (1-2) is exactly 1/2.

1-4. Assume that the face-up card is [♢]:

[♢][?] ··· [?] ··· [?][?][?][?][?] ··· [?][?][?]
[?][?] ··· [?] ··· [?][?][?][?][?] ··· [?][?][?]
⋮
[?][?] ··· [?] ··· [?][?][?][?][?] ··· [?][?][?] .

Its right-hand card must also be a marker. Reveal the markers right next to it one by one. After all the makers on the right side (which are ℓ [♢]s for some ℓ and $n/2$ [♠]s) are face up, reveal the remaining markers on the left side (where the first card's "left" is the last card), namely $(n/2 - \ell - 1)$ [♢]s. For the case where the first card is [♠], we manipulate the sequence of cards similarly to the [♢] case. Note that in this case, we start revealing the markers toward the left side first.

Remove all of the (face-up) n markers.

1-5. After all of the n markers are removed, we regard the first n cards as the value of $\pi(1)$. That is, the pile of these n cards is assigned to player p_1 and corresponds to $\pi(1)$:

p_1 : [?][?] $\cdots$ [?] $\cdots$ [?].

1-6. Similarly, for the remaining cards, repeat steps (1-2)–(1-4) so that we obtain piles corresponding to $\pi(2), \pi(3), \ldots, \pi(n)$.

[Fixed-Point Checking phase]

2-1. To verify that the generated permutation π has no fixed point, arrange the piles of cards assigned to $p_1, p_2, \ldots, p_n$ as below:

p_1 : [?][?] $\cdots$ [?] $\cdots$ [?][?]
p_2 : [?][?] $\cdots$ [?] $\cdots$ [?][?]
$\vdots$
p_n : [?][?] $\cdots$ [?] $\cdots$ [?][?].

2-2. Reveal all the cards on the diagonal to determine if they are all [♣]. If so, π has no fixed point. If one of them is [♡], then the pile corresponds to a fixed point and in this case, we must return to the Random-Permutation Generating phase.

Thus, the first phase of this protocol produces a random permutation $\pi \in S_n$, and then the second phase checks that π has no fixed point. In the first phase, we need to repeat the steps until markers are found, and hence it is a Las Vegas algorithm taking $2n$ trials on average. With respect to the second phase, note that in general, the probability that a random permutation $\pi \in S_n$ has no fixed point is $\sum_{i=0}^{n}(-1)^i/i!$, which is approximately $1/e$, where e is the base of the natural logarithm [3]. Therefore, the average number of how many times we need to execute the Fixed-Point Checking phase is approximately $e \approx 2.7$.

This is the existing protocol for solving the "no fixed point" problem. It uses $2n^2$ cards of four colors, as detailed above. We improve on this efficiency in the succeeding sections.

3 Pile-Scramble Shuffle

In this section, we focus on the process of producing a random permutation and propose an efficient method for achieving this.

Remember that the known protocol [3] uses random cuts and markers to generate a random permutation, as shown in the preceding section. That is, in order to shuffle n piles (each of which consists of n cards and is assigned to a player), we repeatedly apply a random cut to create each value of $\pi(i)$ one by one, while markers are used as "delimiters." Here, instead of using markers, we consider a somewhat more direct way of shuffling piles.

Assume that there are a number of face-down cards that are divided into n piles of the same size. We denote each pile by $pile_i$, $1 \le i \le n$. Given a sequence of piles $(pile_1, pile_2, pile_3, ..., pile_n)$, consider a shuffle operation that outputs $(pile_{\pi(1)}, pile_{\pi(2)}, pile_{\pi(3)}, ..., pile_{\pi(n)})$, where $\pi \in S_n$ is a random permutation. As we now have n piles, a permutation is randomly chosen from the $n!$ possibilities. We call such a shuffling operation a *pile-scramble shuffle*. We believe that the pile-scramble shuffle can be easily implemented by human beings using rubber bands, clips, envelopes, or something similar.

If steps (1-2)–(1-6) in the Random-Permutation Generating phase of the known protocol [3] introduced in Section 2 are replaced with the pile-scramble shuffle, it is obvious that n^2 cards of two colors are sufficient to produce a random permutation. That is, we can generate a random permutation without any marker, meaning that we do not require any trials, and hence can output a random permutation after exactly one pile-scramble shuffle. Therefore, taking the Fixed-Point Checking phase into account, such an improved protocol needs only n^2 cards of two colors and takes an average number of about 2.7 trials to generate a random permutation having no fixed point. Thus, we are able to reduce the numbers of required cards and colors by half (see Table 1 again).

In the next section, we further reduce the number of required cards.

4 Our Main Protocol

In this section, we propose a more efficient method than those mentioned previously. Our main protocol requires only $(2n\lceil\log n\rceil + 6)$ cards to generate a random permutation having no fixed point.

First, in Section 4.1, we show that considering a binary representation of players' indices dramatically reduces the number of required cards. Next, in Section 4.2, we present a sub-protocol to check for fixed points under such a binary representation. Finally, in Section 4.3, by combining these components, we present a complete description of our protocol.

4.1 Binary Representation

In the Crépeau-Kilian protocol [3] presented in Section 2, each player's index $i \in \{1, 2, \ldots, n\}$ and its permuted position $\pi(i)$ are represented by a pile of n

cards, i.e., $(n-1)$ $\boxed{\clubsuit}$s and one $\boxed{\heartsuit}$, say

$$p_i : \overset{1}{\boxed{\clubsuit}}\overset{2}{\boxed{\clubsuit}} \cdots \overset{i}{\boxed{\heartsuit}} \cdots \overset{n}{\boxed{\clubsuit}} \text{ or } \overset{1}{\boxed{\clubsuit}}\overset{2}{\boxed{\clubsuit}} \cdots \overset{\pi(i)}{\boxed{\heartsuit}} \cdots \overset{n}{\boxed{\clubsuit}}.$$

In contrast, we represent this information using a binary representation with $2\lceil \log n \rceil$ cards as follows.

To deal with Boolean values, following the previous studies (e.g., [1, 3, 10, 13]), we use the encoding rule with a pair of cards:

$$\boxed{\clubsuit}\boxed{\heartsuit} = 0, \quad \boxed{\heartsuit}\boxed{\clubsuit} = 1. \tag{1}$$

For a bit $x \in \{0,1\}$, when two face-down cards $\boxed{?}\boxed{?}$ have a value equaling x according to encoding (1) above, the pair of these face-down cards is called a *commitment* to x, and is written as

$$\underbrace{\boxed{?}\boxed{?}}_{x}.$$

Under such an encoding rule, each player's index can be represented by $\lceil \log n \rceil$ commitments, namely $2\lceil \log n \rceil$ cards. Therefore, n players' indices are represented naturally by $2n\lceil \log n \rceil$ cards. Thus, we can greatly reduce the number of required cards to express players' indices.

It is obvious that we can easily produce a random permutation by applying a pile-scramble shuffle (explained in Section 3) to these n piles that are based on this binary expression.

4.2 How to Check for Fixed Points

In this subsection, we present a sub-protocol to check that a random permutation in the form of binary representation has no fixed point.

Assume that a random permutation $\pi \in S_n$ has been generated by a pile-scramble shuffle, as shown in Section 3, based on the binary representation shown in Section 4.1. That is, a pile of $\lceil \log n \rceil$ commitments is assigned to each player p_i:

$$p_i : \underbrace{\boxed{?}\boxed{?}}_{a_{\log n}} \cdots \underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1},$$

where and hereafter, $\log n$ in the subscript means $\lceil \log n \rceil$. Because the pile above corresponds to $\pi(i)$, we have

$$(\pi(i) - 1)_{10} = (a_{\log n} \cdots a_2 a_1)_2.$$

In order to verify that the pile is not a fixed point, namely $\pi(i) \neq i$, we check whether the equation below holds:

$$(a_1 \oplus \overline{b_1}) \wedge (a_2 \oplus \overline{b_2}) \wedge \cdots \wedge (a_{\log n} \oplus \overline{b_{\log n}}) = 0\,, \tag{2}$$

where $\oplus$ denotes the exclusive-or (XOR) operation and bits $b_1, b_2, \cdots, b_{\log n}$ are defined as

$$(i-1)_{10} = (b_{\log n} \cdots b_2 b_1)_2 \ .$$

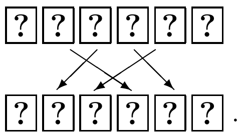
Aiming to compute Eq. (2) efficiently without revealing values a_i, $1 \leq i \leq \lceil \log n \rceil$, we first introduce the existing copy protocol [8], and then present a "one-input-preserving" AND protocol. Finally we describe a sub-protocol for checking that Eq. (2) holds.

Copy Protocol Give a commitment to a bit x together with four additional cards, the known copy protocol [8] generates two copied commitments to x, as follows.

1. Arrange two commitments to 0:

$$\underbrace{\boxed{?}\boxed{?}}_{x}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit} \rightarrow \underbrace{\boxed{?}\boxed{?}}_{x}\underbrace{\boxed{?}\boxed{?}}_{0}\underbrace{\boxed{?}\boxed{?}}_{0}.$$

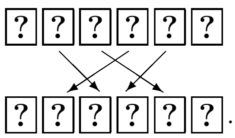
2. Rearrange the order of the sequence as:



3. Bisect the sequence of six cards and switch the two portions randomly (we call this a random bisection cut [8] and denote it by $[\,\cdot\,|\,\cdot\,]$):

$$\left[\boxed{?}\boxed{?}\boxed{?}\,\middle|\,\boxed{?}\boxed{?}\boxed{?}\right] \rightarrow \boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}.$$

4. Rearrange the order of the sequence as:



We then have

$$\underbrace{\boxed{?}\boxed{?}}_{x\oplus r}\underbrace{\boxed{?}\boxed{?}}_{r}\underbrace{\boxed{?}\boxed{?}}_{r},$$

where r is a (uniformly distributed) random bit because of the random bisection cut.

5. Reveal the first two cards from the left. We then have

$$\boxed{\clubsuit}\boxed{\heartsuit}\underbrace{\boxed{?}\boxed{?}}_{x}\underbrace{\boxed{?}\boxed{?}}_{x} \text{ or } \boxed{\heartsuit}\boxed{\clubsuit}\underbrace{\boxed{?}\boxed{?}}_{\bar{x}}\underbrace{\boxed{?}\boxed{?}}_{\bar{x}}.$$

Thus, we obtain two copied commitments to x. In the latter case, we can easily convert $\overline{x}$ to x using the NOT operation that swaps the left and right cards. In addition, the two face-up cards $\boxed{\clubsuit}\boxed{\heartsuit}$ are available for another computation.

One-input-preserving AND Protocol We present a *one-input-preserving AND protocol* that can keep one of input commitments after the AND computation. The protocol can be constructed immediately based on two known ideas: the AND protocol [8] and the half-adder protocol [6].

First, we present some notation. For a pair of bits (x, y), define operations get and shift as

$$\mathsf{get}^0(x, y) = x; \quad \mathsf{get}^1(x, y) = y,$$
$$\mathsf{shift}^0(x, y) = (x, y); \quad \mathsf{shift}^1(x, y) = (y, x)\,.$$

Note that

$$a \wedge b = \mathsf{get}^{a \oplus r}(\mathsf{shift}^r(0, b)) \tag{3}$$

for an arbitrary bit $r \in \{0, 1\}$. In addition, for two bits x and y, the expression

$$\underbrace{\boxed{?}\boxed{?}\boxed{?}\boxed{?}}_{(x,y)}$$

means

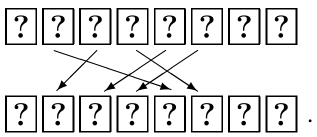
$$\underbrace{\boxed{?}\boxed{?}}_{x}\underbrace{\boxed{?}\boxed{?}}_{y}\,.$$

The following is a one-input-preserving AND protocol that produces not only a commitment to $a \wedge b$ but also a commitment to the input a using eight cards.

1. In addition to the input commitments to a and b, arrange two commitments to 0 as follows:

$$\underbrace{\boxed{?}\boxed{?}}_{a}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\underbrace{\boxed{?}\boxed{?}}_{b} \rightarrow \underbrace{\boxed{?}\boxed{?}}_{a}\underbrace{\boxed{?}\boxed{?}}_{0}\underbrace{\boxed{?}\boxed{?}}_{0}\underbrace{\boxed{?}\boxed{?}}_{b}\,.$$

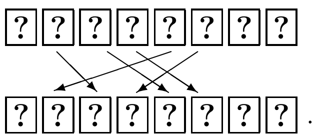
2. Rearrange the order of the sequence as:



3. Apply a random bisection cut:

$$\left[\boxed{?}\boxed{?}\boxed{?}\boxed{?}\middle|\boxed{?}\boxed{?}\boxed{?}\boxed{?}\right] \rightarrow \boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\,.$$

4. Rearrange the order of the sequence as:



We then have

$$\underbrace{\boxed{?}\boxed{?}}_{a\oplus r}\underbrace{\boxed{?}\boxed{?}}_{r}\underbrace{\boxed{?}\boxed{?}\boxed{?}\boxed{?}}_{\mathsf{shift}^r(0,b)},$$

where r is a (uniformly distributed) random bit.

5. Reveal the first two cards. If they are $\boxed{\clubsuit}\boxed{\heartsuit}$, we have $a \oplus r = 0$, i.e., $r = a$. Therefore, the output is (see Eq. (3)):

$$\boxed{\clubsuit}\boxed{\heartsuit}\underbrace{\boxed{?}\boxed{?}}_{a}\underbrace{\boxed{?}\boxed{?}}_{a\wedge b}\boxed{?}\boxed{?}\ .$$

If they are $\boxed{\heartsuit}\boxed{\clubsuit}$, we have $a \oplus r = 1$, i.e., $r = \bar{a}$. Therefore, the output is:

$$\boxed{\heartsuit}\boxed{\clubsuit}\underbrace{\boxed{?}\boxed{?}}_{\overline{a}}\boxed{?}\boxed{?}\underbrace{\boxed{?}\boxed{?}}_{a\wedge b}\ .$$

In this way, we can obtain commitments to both $a \wedge b$ and a. The two face-up cards $\boxed{\clubsuit}\boxed{\heartsuit}$ are still available for another computation. In addition, the two cards of the remaining commitment can also be available after they are shuffled.

Sub-protocol for Checking Eq. (2) Given the discussion above, we are ready to present a procedure for checking Eq. (2) to determine if there are fixed points. Given a pile

$$p_i : \underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1},$$

the following sub-protocol computes the value of

$$(a_1 \oplus \overline{b_1}) \wedge (a_2 \oplus \overline{b_2}) \wedge \cdots \wedge (a_{\log n} \oplus \overline{b_{\log n}})\,,$$

where

$$(i-1)_{10} = (b_{\log n} \cdots b_2 b_1)_2\ .$$

1. Arrange $\lceil \log n \rceil$ input commitments and six additional cards as follows:

$$\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_3}\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\ .$$

2. Copy the commitment to a_1 using the copy protocol [8] mentioned above:

$$\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_3}\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\ .$$

3. Apply the NOT computation depending on the values of b_1 and b_2 so that we have
$$\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_3}\underbrace{\boxed{?}\boxed{?}}_{a_2\oplus\overline{b_2}}\underbrace{\boxed{?}\boxed{?}}_{a_1\oplus\overline{b_1}}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\,.$$
Note that each value of b_i is public.
4. Apply the one-input-preserving AND protocol presented above to obtain commitments to $(a_1 \oplus \overline{b_1}) \wedge (a_2 \oplus \overline{b_2})$ and $(a_2 \oplus \overline{b_2})$. Furthermore, apply the NOT computation to the latter commitment depending on the value of b_2. We then have
$$\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_3}\quad\underbrace{\boxed{?}\boxed{?}}_{(a_1\oplus\overline{b_1})\wedge(a_2\oplus\overline{b_2})}\quad\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\,.$$
5. Similarly, obtain commitments to $(a_1 \oplus \overline{b_1}) \wedge (a_2 \oplus \overline{b_2}) \wedge (a_3 \oplus \overline{b_3})$ and a_3:
$$\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\quad\underbrace{\boxed{?}\boxed{?}}_{(a_1\oplus\overline{b_1})\wedge(a_2\oplus\overline{b_2})\wedge(a_3\oplus\overline{b_3})}\quad\underbrace{\boxed{?}\boxed{?}}_{a_3}\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\,.$$
Repeat this until we have
$$\underbrace{\boxed{?}\boxed{?}}_{(a_1\oplus\overline{b_1})\wedge(a_2\oplus\overline{b_2})\wedge\cdots\wedge(a_{\log n}\oplus\overline{b_{\log n}})}\quad\underbrace{\boxed{?}\boxed{?}}_{a_{\log n}}\cdots\underbrace{\boxed{?}\boxed{?}}_{a_3}\underbrace{\boxed{?}\boxed{?}}_{a_2}\underbrace{\boxed{?}\boxed{?}}_{a_1}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\,.$$
6. Reveal the commitment to $(a_1 \oplus \overline{b_1}) \wedge (a_2 \oplus \overline{b_2}) \wedge \cdots \wedge (a_{\log n} \oplus \overline{b_{\log n}})$. If the value is 1, then this is a fixed point. Otherwise, it is not a fixed point. It should be noted that in either case, any commitments to $a_1, a_2, \ldots, a_{\log n}$ are not lost.

4.3 Description of Our Proposed Protocol

We are now ready to present an efficient protocol for generating a random permutation having no fixed point. Our protocol uses $(2n\lceil\log n\rceil + 6)$ cards to produce n piles corresponding to this random permutation.

1. Using $n\lceil\log n\rceil$ $\boxed{\clubsuit}$s and $n\lceil\log n\rceil$ $\boxed{\heartsuit}$s, arrange $n\lceil\log n\rceil$ commitments according to players' indices based on the binary representation:
$$p_1 : \underbrace{\boxed{?}\boxed{?}}_{0}\cdots\underbrace{\boxed{?}\boxed{?}}_{0}\underbrace{\boxed{?}\boxed{?}}_{0}$$
$$p_2 : \underbrace{\boxed{?}\boxed{?}}_{0}\cdots\underbrace{\boxed{?}\boxed{?}}_{0}\underbrace{\boxed{?}\boxed{?}}_{1}$$
$$\vdots$$
$$p_n : \underbrace{\boxed{?}\boxed{?}}_{1}\cdots\underbrace{\boxed{?}\boxed{?}}_{1}\underbrace{\boxed{?}\boxed{?}}_{1}\,.$$

2. Regarding each row as a pile, apply a pile-scramble shuffle to the n piles; we then obtain a random permutation π in which the i-th pile corresponds to $\pi(i)$:

$$\begin{array}{l} p_1 : \boxed{?}\boxed{?}\cdots\boxed{?}\boxed{?}\boxed{?}\boxed{?} \\ p_2 : \boxed{?}\boxed{?}\cdots\boxed{?}\boxed{?}\boxed{?}\boxed{?} \\ \qquad\qquad\vdots \\ p_n : \boxed{?}\boxed{?}\cdots\boxed{?}\boxed{?}\boxed{?}\boxed{?}. \end{array}$$

3. Using six additional cards, apply the sub-protocol presented in Section 4.2 to confirm that π has no fixed point, that is, to verify that p_i is not a fixed point for every i, $1 \leq i \leq n$, in turns. If we find a fixed point, then we go back to step (2). If we confirm that there is no fixed point, the permutation π is a desired one.

This is our main protocol for solving the “no fixed point” problem with $O(n \log n)$ cards.

5 Conclusions

The known protocol [3] requires $2n^2$ cards of four colors to generate a random permutation having no fixed point. In this paper, we first devised a new shuffle operation called a pile-scramble shuffle that immediately enabled us to achieve the same task using only n^2 cards of two colors. Furthermore, we showed that using a binary representation dramatically reduces the number of required cards, that is, $(2n\lceil \log n \rceil + 6)$ cards of two colors are sufficient.

In our protocol, the $2n\lceil \log n \rceil$ cards are used to hold each players' index, and the remaining six cards correspond to the additional cards $\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}\boxed{\clubsuit}\boxed{\heartsuit}$ required to execute the sub-protocol for checking fixed points. This comes from the fact that the one-input-preserving AND protocol given in Section 4.2 requires four additional cards. Recently, it was shown that such a one-input-preserving AND computation can be done with only two additional cards [11]. Therefore, applying this recently invented protocol [11], we can reduce the number of required cards to $2n\lceil \log n \rceil + 4$.

In addition to the protocol solving the “no fixed point” problem, Crépeau and Kilian designed a general protocol for producing a random permutation that satisfies a predetermined condition such as having no short cycle of length at most k, and showed that it can be applied to the “Discreet Solitary Games” [3]. Thus, it is intriguing future work to design an efficient way to determine whether a given permutation based on our binary representation has k-cycles.

Although the card-based protocol is an unconventional way to secure multi-party computations, this approach has many advantages. The most important feature is that even nonspecialists are able to easily understand why the computation is secure.

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